Changing Sampling rate: Decimation and Interpolation

Digital Interpolation by a Factor of *L*

Applications of digital interpolators include sampling rate conversion in multi-rate communication systems and up-sampling for improved graphical representation. To change a sampling rate by a factor of $V=L/D$ (where L and D are integers), the signal is first interpolated by a factor of L, and then decimated by a factor of D. Before decimation the signal must be filtered by a lowpass filter which acts as an interpolator and also as an anti-ailiasing filter for decimation. The cutoff frequency of lowpass filter should be set to $\min(F_s/2L, F_s/2D)$.

Consider a band-limited discrete-time signal $x(m)$ with a base-band spectrum $X(f)$ as shown in Figure 1. The sampling rate can be increased by a factor of L through interpolation of L–1 samples between every two samples of $x(m)$. In the following it is shown that digital interpolation by a factor of L can be achieved through a two-stage process of:

(a) Insertion of L–1 zeros in between every two samples and
(b) Low-pass filtering of the zero-inserted signal by a filter with a cutoff frequency of $F_s/2L$, where F_s is the sampling rate.

Consider the zero-inserted signal $x_z(m)$ obtained by inserting L–1 zeros between every two samples of $x(m)$ and expressed as

$$x_z(m) = \begin{cases} x\left(\dfrac{m}{L}\right), & m = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The spectrum of the zero-inserted signal is related to the spectrum of the original discrete-time signal by

$$\begin{aligned} X_z(f) &= \sum_{m=-\infty}^{\infty} x_z(m) e^{-j2\pi fm} \\ &= \sum_{m=-\infty}^{\infty} x(m) e^{-j2\pi fLm} \\ &= X(L.f) \end{aligned} \tag{2}$$

Equation (2) states that the spectrum of the zero-inserted signal $X_z(f)$ is a frequency-scaled version of the spectrum of the original signal $X(f)$. Figure 1 shows that the base-band spectrum of the zero-inserted signal is composed of L repetitions of the based band spectrum of the original signal. The interpolation of the zero-inserted signal is therefore equivalent to filtering out the repetitions of $X(f)$ in the base band of $X_z(f)$, as illustrated in 1. Note that to maintain the real-time duration of the signal the sampling rate of the interpolated signal $x_z(m)$ needs to be increased by a factor of L.

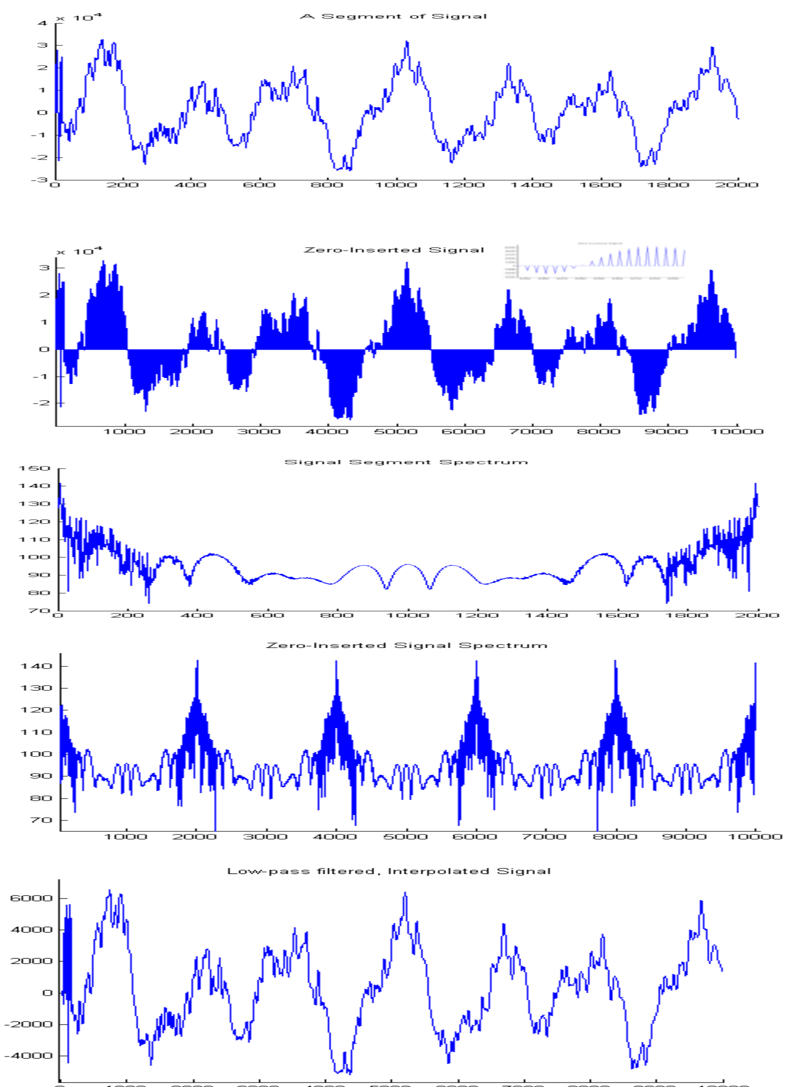
Interpolation Using DFT

Taking the DFT of N samples of zero-inserted signal we have

$$\begin{aligned} X_z(k) &= \sum_{m=0}^{NL-1} x_z(m) e^{-j\frac{2\pi}{NL}mk} \\ &= \sum_{m=0}^{N} x(m) e^{-j\frac{2\pi}{NL}mkN} \qquad k=0,1,, \ldots, NL-1 \qquad (3) \\ &= X(k) \end{aligned}$$

Note from this equation that within a frequency interval of to 2π or (0 to F_S Hz) there are L repetition of the spectrum of X(k), this is because stretching the signal by a factor of L shrinks its spectrum by a factor of 1/L.

(a)
(b)
(c)
(d)
(e)



Figure(1) - (a) Original signal, (b) zero-insetred signal, (c) spectrum of original signal, (d) spectrum of zero-inserted signal, (e) interpolated signal

Digital Decimation by a Factor of *L*

Consider a band-limited discrete-time signal $x(m)$ with a base-band spectrum $X(f)$. The sampling rate can be decreased by a factor of L through discarding of L–1 samples for every L samples of $x(m)$. In the following it is shown that digital decimation by a factor of L can be achieved through a two-stage process of:

(a) Low-pass filtering of the zero-inserted signal by a filter with a cutoff frequency of $F_s/2L$, where F_s is the sampling rate. This is the anti-alisaing process.
(b) Discarding of L–1 samples for every L samples.

Mathematical model of decimation. Consider a resampled signal $x_r(m)$ expressed as the product of the orginal sample $x(m)$ and the resampling pulse train $p(m)$ as

$$x_r(m) = x(m).p(m) = \sum_{k=-\infty}^{\infty} x(k)\delta(k-Lm) = \begin{cases} x(Lm), & m = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where the pulse train $p(m)$ is given by

$$p(m) = \sum_{k=-\infty}^{\infty} \delta(k-Lm) \quad \Leftrightarrow \quad P(f) = \sum_{k=-\infty}^{\infty} \delta\left(f - k\frac{F_s}{L}\right) \tag{5}$$

Note as the distance between pulses increase by a factor of L the distance between their frequency components shrink by a factor of $1/L$.

From the Fourier properties, the spectrum of a resampled signal is the convolution of the spectra of the $x(m)$ and $p(m)$

$$X_r(f) = \sum_{k=-\infty}^{\infty} X\left(f + k\frac{F_s}{L}\right) \tag{6}$$

The down-sampled signal can be obtained from the resampled signal as

$$x_d(m) = x_r(mL) \tag{7}$$

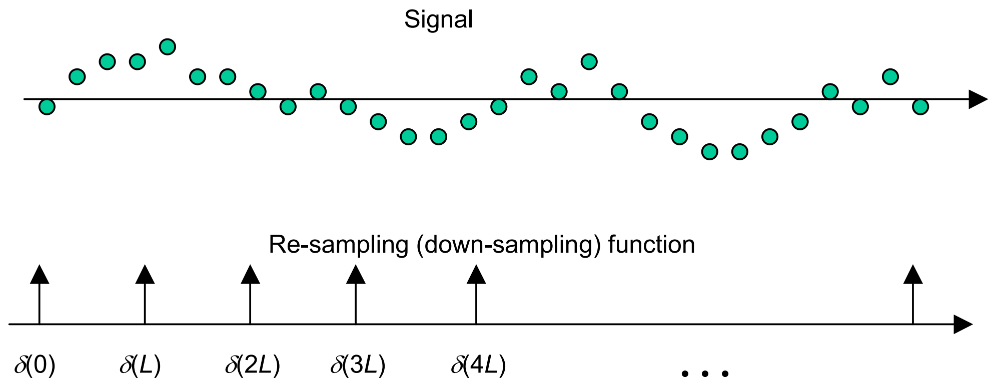


Figure2 Resampling process.

The spectrum of the decimated signal can be related to the spectrum of the resampled discrete-time signal as follows

$$
\begin{aligned}
X_d(f) &= \sum_{m=0}^{N-1} x_d(m) e^{-j2\pi f m} \\
&= \sum_{m=-\infty}^{\infty} x_r(mL) e^{-j2\pi \frac{f}{L} mL} \\
&= \sum_{m=-\infty}^{\infty} x_r(m) e^{-j2\pi \frac{f}{L} m} \\
&= X_r(f/L)
\end{aligned}
\qquad (8)
$$

Note that in developing the third line of Eq(8) from the second line we used the fact that inbetween every two signal samples of $x_r(m)$ there are L-1 zeros.

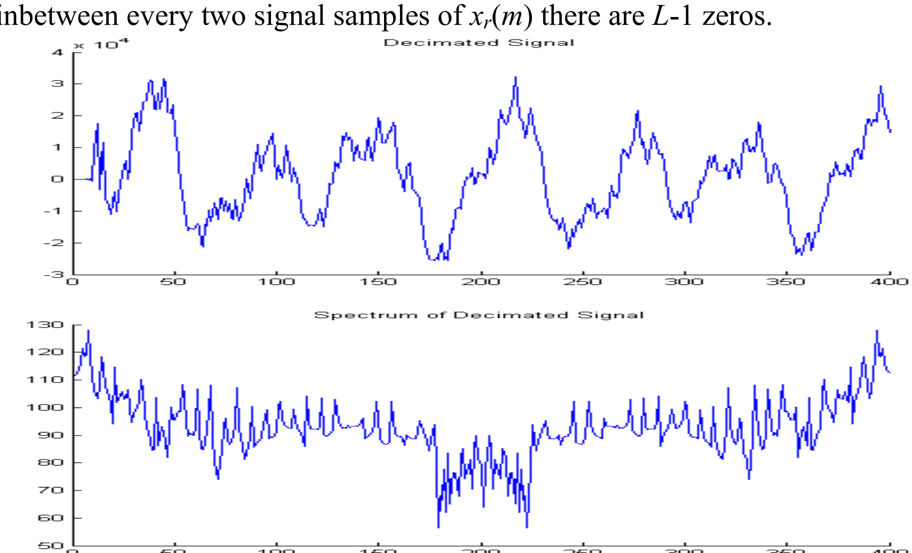


Figure 3 (a) Decimated signal and (b) its spectrum.